

# $f$ -vectors implying vertex decomposability

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**ABSTRACT.** We prove that if a pure simplicial complex  $\Delta$  of dimension  $d$  with  $n$  facets has the least possible number of  $(d-1)$ -dimensional faces among all complexes with  $n$  faces of dimension  $d$ , then it is vertex decomposable. This answers a question of J. Herzog and T. Hibi. In fact we prove a generalization of their theorem using combinatorial methods.

## 1. Introduction

We call a simplicial complex *pure* if all its facets are of the same dimension.

**DEFINITION 1.** *A pure simplicial complex  $\Delta$  of dimension  $d$  and  $n$  facets is called extremal if it has the least possible number of  $(d-1)$ -dimensional faces among all complexes with  $n$  faces of dimension  $d$ .*

In particular, for  $d = 0$  all zero dimensional complexes are extremal, since all of them have exactly one  $(-1)$ -dimensional face, namely the empty set.

In this paper we generalize and prove by only combinatorial means the following theorem of Herzog and Hibi from 1999.

**THEOREM 1.** ([8], Theorem 2.3) *An extremal simplicial complex is Cohen-Macaulay over an arbitrary field.*

Their proof is algebraic and uses results from [1] and [6]. In fact they asked for a combinatorial proof. We give it by proving that an extremal simplicial complex is vertex decomposable. It is well-known that vertex decomposable complexes are Cohen-Macaulay. Our proof goes along the lines of the proof of Kruskal-Katona inequality. We start with a presentation of some necessary preliminaries.

**1.1. Vertex decomposable and Cohen-Macaulay complexes.** For a simplex  $\sigma$  in a simplicial complex  $\Delta$ , the simplicial complex

$$\{\tau \in \Delta : \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$$

is called a *link* of  $\sigma$  in  $\Delta$ , and denoted by  $\text{lk}_\Delta \sigma$ . For a vertex  $x$  of  $\Delta$  by  $\Delta \setminus x$  we mean the simplicial complex  $\{\tau \in \Delta : x \notin \tau\}$ .

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DEFINITION 2. A pure simplicial complex  $\Delta$  is vertex decomposable if one of the following holds:

- (1)  $\Delta$  is empty,
- (2)  $\Delta$  is a single vertex,
- (3) for some vertex  $x$  both  $\text{lk}_\Delta\{x\}$  and  $\Delta \setminus x$  are pure and vertex decomposable.

DEFINITION 3. For a simplicial complex  $\Delta$  on the set of vertices  $\{1, \dots, n\}$  and a given field  $\mathbb{K}$ , the Stanley-Reisner ring (the face ring) is  $\mathbb{K}[\Delta] := \mathbb{K}[x_1, \dots, x_n]/I$ , where  $I$  is generated by all square-free monomials  $x_{i_1} \cdots x_{i_l}$  for which  $\{i_1, \dots, i_l\}$  is not a face in  $\Delta$ .

When we say that a simplicial complex is *Cohen-Macaulay* we always mean that its Stanley-Reisner ring has this property.

The following is a folklore result (we refer the reader to e.g. [2]).

THEOREM 2. For a simplicial complex  $\Delta$  the following implications hold:

$\Delta$  is vertex decomposable  $\Rightarrow \Delta$  is shellable  $\Rightarrow \Delta$  is Cohen-Macaulay over any field.

We recall a combinatorial description of Cohen-Macaulay complexes.

THEOREM 3. ([13]) Let  $R = \mathbb{K}[\Delta]$  be the face ring of  $\Delta$ . Then the following conditions are equivalent:

- (1)  $R$  is Cohen-Macaulay ring,
- (2)  $\tilde{H}_i(\text{lk}_\Delta \sigma; \mathbb{K}) = 0$  for  $i < \dim(\text{lk}_\Delta \sigma)$  for all simplices  $\sigma \in \Delta$ .

For classical techniques of counting homologies we refer the reader to [7], [14]. For entertaining ones we advise Section 3.2 of [12].

**1.2. Kruskal-Katona theorem.** One of the most natural questions concerning simplicial complexes is:

What is the minimum number of  $(k-1)$ -element faces in simplicial complex with  $n$  faces of size  $k$ ?

This question was answered independently by Kruskal [11] and Katona [10] in 1960's. For a positive integer  $k$ , they enlisted all  $k$ -element subsets of integers in the following order, called the *squashed order*:  $A < B$  if  $\max(A \setminus B) < \max(B \setminus A)$ . Let  $S_k(n)$  be the set of first  $n$  sets in this list. For a given set  $\mathcal{U}$  of  $k$ -element sets, denote by  $\Delta\mathcal{U}$  the set of all  $(k-1)$ -element sets which are contained in some member of  $\mathcal{U}$ . The Kruskal-Katona theorem reads as follows.

THEOREM 4. For a positive integers  $n, k$  and a set  $\mathcal{U}$  of  $n$  sets of size  $k$  we have

$$|\Delta\mathcal{U}| \geq |\Delta S_k(n)|.$$

This result was further generalized by Clements and Lindström in [3]. Daykin [4, 5] gave two simple proofs, and later Hilton [9] gave another one. For an algebraic proof we refer the reader to [1]. We will work mainly with Hilton's idea.

Note that the cardinality of  $\Delta S_k(n)$  may be easily determined. For a given  $k$ , each positive integer  $n$  can be uniquely expressed as

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

with  $1 \leq t \leq a_t$  and  $a_t < \cdots < a_k$ . We have

$$\delta_{k-1}(n) := |\Delta S_k(n)| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1}.$$

As a consequence of Kruskal-Katona theorem we get:

**COROLLARY 1.** *A pure simplicial complex  $\Delta$  of dimension  $d > 0$  with  $f$ -vector  $(f_0, \dots, f_d)$  is extremal if and only if  $f_{d-1} = \delta_d(f_d)$ .*

## 2. The main result

For a better understanding of the assumption that  $\Delta$  is extremal we will use Hilton's idea from his proof [9] of the Kruskal-Katona theorem. First we define sets similar to  $S_k(n)$ . Let  $S_k^i(n)$  denote the first  $n$  sets of  $k$ -element subsets of integers in the squashed order ( $A < B$  if  $\max(A \setminus B) < \max(B \setminus A)$ ) which do not contain  $i$ . We also denote by  $\{i\}(\cup)\mathfrak{U}$  the set  $\{\{i\} \cup A : A \in \mathfrak{U}\}$ .

Let  $\mathfrak{U}$  be a  $n$ -element set of  $k$ -element sets, let  $V = \bigcup_{A \in \mathfrak{U}} A$  be an underlying set, and let  $v$  be its cardinality. For  $i \in V$ , let  $B_i = \{A \in \mathfrak{U} : i \notin A\}$ ,  $C_i = \{A \setminus \{i\} : i \in A \in \mathfrak{U}\}$ , and let  $b_i, c_i$  be the respective cardinalities. Note that  $c_i \neq 0$ . We want to find an index  $i$  such that  $|\Delta B_i| > |C_i|$ .

**LEMMA 1.** *Either there exists an  $i$  such that  $|\Delta B_i| > |C_i|$ , or  $\mathfrak{U}$  consists of all possible  $k$ -element subsets of  $V$ .*

**PROOF.** We are going to count the sum of cardinalities of both sets when  $i$  runs over all elements of  $V$ . Then

$$\sum_{i \in V} |\Delta B_i| \geq kn(v-k)/(v-k) = kn = \sum_{i \in V} |C_i|,$$

since at left hand side each  $A \in \mathfrak{U}$  gives  $k$  distinct sets in its boundary, and it is counted once for every  $i \notin A$ . Some sets in boundaries of sets from  $B_i$  can be the same, but their number is at most  $(v-1) - (k-1) = v-k$ . On the other hand each  $A \in \mathfrak{U}$  is counted  $k$  times at the right side. Hence we can find a desired  $i$ , or the above bounds are tight. In the latter case, when  $A \in \Delta B_i$ , all  $v-k$  possibilities of completing it to a  $k$ -element set has to be in  $\mathfrak{U}$ . This means that  $\mathfrak{U}$  consists of all possible  $k$ -element subsets of  $V$  because from any set in  $\mathfrak{U}$  we can delete any element and insert any other.  $\square$

**LEMMA 2.** *If  $\Delta$  is an extremal simplicial complex of positive dimension, then there exists a vertex  $x$  such that both  $\text{lk}_\Delta\{x\}$  and  $\Delta \setminus x$  are extremal.*

**PROOF.** Let  $\Delta$  be of dimension  $d-1 > 0$  and let  $\mathfrak{U}$  be the set of all  $d$ -element sets in  $\Delta$ . If  $\mathfrak{U}$  consists of all possible  $d$ -element subsets of a given  $v$ -element set, then the assertion of the lemma is clearly true (we can take any vertex). Otherwise, due to Lemma 1, there exists an  $i \in V$  such that  $|\Delta B_i| > |C_i|$ . We have that

$$\Delta \mathfrak{U} = \Delta B_i \cup C_i \cup (\{i\}(\cup)\Delta C_i).$$

Since  $\Delta B_i$  and  $\{i\}(\cup)\Delta C_i$  are disjoint, it follows that

$$|\Delta \mathfrak{U}| \geq |\Delta B_i| + |\{i\}(\cup)\Delta C_i| > |C_i| + |\{i\}(\cup)\Delta C_i|.$$

So, by Theorem 4,

$$(2.1) \quad |\Delta \mathfrak{U}| \geq |\Delta S_d^i(b_i)| + |\{i\}(\cup)\Delta S_{d-1}^i(c_i)|,$$

and

$$(2.2) \quad |\Delta \mathfrak{U}| > |S_{d-1}^i(c_i)| + |\{i\}(\cup)\Delta S_{d-1}^i(c_i)|.$$

Since  $\Delta S_d^i(b_i) = S_{d-1}^i(e)$  for some  $e$ , there are now two possibilities:

- (1)  $\Delta S_d^i(b_i) \subset S_{d-1}^i(c_i)$ , then by (2.2) we get
$$|\Delta \mathfrak{U}| > |S_{d-1}^i(c_i)| + |\{i\}(\cup) \Delta S_{d-1}^i(c_i)| = |\Delta(S_d^i(b_i) \cup (\{i\}(\cup) S_{d-1}^i(c_i)))|,$$
which contradicts the assumption that  $\Delta$  is extremal, since a complex generated by sets  $S_d^i(b_i) \cup (\{i\}(\cup) S_{d-1}^i(c_i))$  has  $b_i + c_i = |\mathfrak{U}|$  maximal faces.
- (2)  $\Delta S_d^i(b_i) \supset S_{d-1}^i(c_i)$ , then by (2.1) we obtain
$$|\Delta \mathfrak{U}| \geq |\Delta S_d^i(b_i)| + |\{i\}(\cup) \Delta S_{d-1}^i(c_i)| = |\Delta(S_d^i(b_i) \cup (\{i\}(\cup) S_{d-1}^i(c_i)))|,$$
and equality holds if and only if  $C_i \subset \Delta B_i$ ,  $|\Delta B_i| = |\Delta S_d^i(b_i)|$ , and  $|\Delta C_i| = |\{i\}(\cup) \Delta S_{d-1}^i(c_i)| = |\Delta S_{d-1}^i(c_i)|$ .

The complex  $\Delta$  is extremal, so equality holds, and we get that  $C_i \subset \Delta B_i$  and  $[B_i], [C_i]$  are extremal, where  $[A]$  means the simplicial complex generated by the set of faces  $A$ . Observe that  $\text{lk}_\Delta \{i\} = [C_i]$  and  $\Delta \setminus i = [B_i]$ . The first equality is obvious, while the second is not as clear. If  $\sigma = \{v_1, \dots, v_k\}$  is a face in  $\Delta \setminus i$  then it is a subface of some facet  $F = \{v_1, \dots, v_d\}$ . If  $i$  does not belong to  $F$  then  $F \in [B_i]$  and so  $\sigma$  does. Otherwise,  $F \setminus \{i\} \in C_i \subset \Delta B_i$ , so  $F \setminus \{i\} \cup \{j\} \in B_i$  for some  $j$ . Hence  $\sigma \in [B_i]$ . Now  $i = x$  gives the assertion.  $\square$

Finally, we are ready to prove the generalization of Theorem 1.

**THEOREM 5.** *An extremal simplicial complex is vertex decomposable.*

**PROOF.** The proof goes by an induction on  $d$  the dimension of  $\Delta$  and secondly on the number of facets. If  $d = 0$  then  $\Delta$  consists of points and by the definition it is vertex decomposable. When  $d > 0$ , then by Lemma 2 there exists a vertex  $x$ , such that both complexes  $\text{lk}_\Delta \{x\}$  and  $\Delta \setminus x$  are extremal. The first is of lower dimension; and the second either has the same dimension as  $\Delta$  but fewer facets, or it has smaller dimension. By the inductive hypothesis, both  $\text{lk}_\Delta \{x\}$  and  $\Delta \setminus x$  are vertex decomposable, and as a consequence  $\Delta$  also is.  $\square$

The above result is best possible in the following sense. Let  $\Delta$  be a pure simplicial complex of dimension  $d > 0$  with  $f$ -vector  $(f_0, \dots, f_d)$ , and with  $f_{d-1} = \delta_d(f_d) + c$ ,  $c \in \mathbb{N}$ . Due to Corollary 1 the meaning of Theorem 5 is that if  $c = 0$ , then  $\Delta$  is vertex decomposable. But even for  $c = 1$  complex  $\Delta$  does not have to be Cohen-Macaulay, which by Theorem 2 is a weaker property than vertex decomposability. We show the following.

**EXAMPLE 1.** *We have that  $\delta_d(2) = 2d+1$ . Let  $\Delta$  be a pure simplicial complex of dimension  $d$  with the set of facets  $\mathfrak{U}$  consisting of two disjoint ones. Then  $|\Delta \mathfrak{U}| = 2d + 2$ , and  $\tilde{H}_0(\text{lk}_\Delta \emptyset; \mathbb{K}) = 1$ , so due to Theorem 3 complex  $\Delta$  is not Cohen-Macaulay over any field  $\mathbb{K}$ , and as a consequence it is also not vertex decomposable.*

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